Pre-Semester Prep Workshop Series

These interactive workshops will review all foundational material leading up to the specified course so you are better equipped to hit the ground running.

Synchronous in-person in the ESS suite & virtual via Zoom

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+Attend & give feedback for Physics I, Pre-Calc/Trip, Calc 1, or Calc 2 sessions for access to a general knowledge exam.

RSVP is preferred but not required

ess.unm.edu/events > January

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Presented by:
Steven Kao
Michelle Maestas

Calculus 2 Prep
(Calc 1 Review)
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1. Numerical and graphical explanation of limit
2. Formal definition of limit, limit properties, limit at infinity, L'Hopital's Rule
3. Using limits to determine vertical, horizontal, and slant asymptotes of rational functions
4. Determining whether a function is continuous
5. Definition of derivative, linear properties of the derivative
6. Product rule, quotient rule, and chain rule
7. Derivative of trig and exponential functions
8. Derivative as a rate of change, implicit differentiation
Topics to be covered (Continued):

9. Related rates
10. Maxima and minima, Mean Value Theorem
11. Using the 1st and 2nd derivatives to graph the original function
12. Optimization, Extreme Value Theorem, linear approximation using differentials
13. Reimann sums, definite integrals, Fundamental Theorem of Calculus 1 and 2
14. Properties of integrals, substitution
15. Area between curves, volume of solids of revolution, length of curves
Limits
Limits: An Informal Definition

The limit is taken as the input, \( x \), approaches a specific value from either side. In this case, \( x \) is approaching \( c \).

\[
\lim_{x \to c} f(x) = L
\]

The limit itself is a value of the function, i.e., its \( y \) value. In this case, the limit is \( L \).

SO...
If \( f(x) \) becomes arbitrarily close to a single number, \( L \), as \( x \) approaches \( c \) from either side, then the limit of \( f(x) \) as \( x \) approaches \( c \) is \( L \).
Limits: Formal Definition

The $\varepsilon-\delta$ definition of a limit: $\varepsilon$ (epsilon) and $\delta$ (delta) are small positive numbers.

- $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$
  - i.e., $|f(x) - L| < \varepsilon$.

- $x$ lies in the interval $(c - \delta, c + \delta)$
  - i.e., $0 < |x - c| < \delta$.
  - $\uparrow$ means $x \neq c$

If $f(x)$ becomes arbitrarily close to a single number, $L$, as $x$ approaches $c$ from either side, then the limit of $f(x)$ as $x$ approaches $c$ is $L$.

**FORMAL DEFINITION:**
Let $f$ be a function on an open interval containing $c$ (except possibly at $c$) and let $L$ be a real number. The statement

$$\lim_{{x \to c}} f(x) = L$$

means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

This formal definition is not used to evaluate limits. Instead, it is used to prove other theorems.
THEOREM 1—Limit Laws  

If $L$, $M$, $c$, and $k$ are real numbers and

\[ \lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \]

then

1. **Sum Rule**: \[ \lim_{x \to c} (f(x) + g(x)) = L + M \]
2. **Difference Rule**: \[ \lim_{x \to c} (f(x) - g(x)) = L - M \]
3. **Constant Multiple Rule**: \[ \lim_{x \to c} (k \cdot f(x)) = k \cdot L \]
4. **Product Rule**: \[ \lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M \]
5. **Quotient Rule**: \[ \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0 \]
6. **Power Rule**: \[ \lim_{x \to c} [f(x)]^n = L^n, \quad n \text{ a positive integer} \]
7. **Root Rule**: \[ \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer} \]

(If $n$ is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)
L’Hospital’s Rule

Let $f$ and $g$ be differentiable functions where $g'(x) \neq 0$ near $x = a$ (except possible at $x = a$). If

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

produces the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{\infty}{0}$, $\frac{-\infty}{\infty}$, $\frac{\infty}{-\infty}$, or $\frac{-\infty}{0}$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit exists.
L'Hôpital’s Continued:

Once you have the functions \( f(x) \) and \( g(x) \) in an indeterminant form, you will then take the derivative of numerator and denominator until you have an exact limit value.

\[
\lim_{{x \to 0}} \frac{\sin x}{{x}} = \frac{0}{0}
\]

Using L'Hôpital’s Rule:

\[
\lim_{{x \to 0}} \frac{\sin x}{{x}} = \lim_{{x \to 0}} \frac{\cos x}{1}
\]

\[
\lim_{{x \to 0}} \cos x = 1 \quad \text{and} \quad \lim_{{x \to 0}} 1 = 1
\]

\[
\lim_{{x \to 0}} \frac{\sin x}{{x}} = \frac{1}{1}
\]
Practice!

a. \[ \lim_{x \to 2} (x^2 - 4) \]

b. \[ \lim_{x \to 0} \frac{x^3 - 4x}{2x^2 + 3x} \]

c. \[ \lim_{x \to -1} \frac{(x+1)^2(x-1)}{x^3 + 1} \]

d. \[ \lim_{x \to 0} \frac{x^3 - 2x^2 + x}{2x^3 + x^2 - 2x} \]
Answers:

a) 0
b) $-\frac{4}{3}$
c) 0
d) $-\frac{1}{2}$
Limits to Find Asymptotes
Limits and Asymptotes

$x = a$ is a vertical asymptote of $f(x)$ if at least one of the following is true.

\[
\begin{align*}
\lim_{{x \to a}} f(x) &= \infty \\
\lim_{{x \to a^+}} f(x) &= \infty \\
\lim_{{x \to a^-}} f(x) &= \infty
\end{align*}
\]

\[
\begin{align*}
\lim_{{x \to a}} f(x) &= -\infty \\
\lim_{{x \to a^+}} f(x) &= -\infty \\
\lim_{{x \to a^-}} f(x) &= -\infty
\end{align*}
\]

$y = L$ is a horizontal asymptote of $y = f(x)$ if either of the following is true.

\[
\begin{align*}
\lim_{{x \to \infty}} f(x) &= L \\
\lim_{{x \to -\infty}} f(x) &= L
\end{align*}
\]
Practice this Idea:

\[ \lim_{x \to 1} \frac{x^3}{(x+1)^2} \]
Vertical Asymptotes

• Vertical asymptotes occur when a function is undefined (usually the zeros of the denom.)

• At a vertical asymptotes, the limit of the function is +/- infinity (does not exist)

• At the asymptote, the function increases or decreases without bound.

• Also, \[ \lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x) \]
**Definition: Horizontal Asymptote**

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

$x = 2$, the vertical asymptote

$y = 4$, the horizontal asymptote

$$y = \frac{1}{x - 2} + 4$$
Ex: Find any H.A. of the graph of \( f(x) = \frac{x}{\sqrt{x^2 + 1}} \)

Investigate with both a graph and a table...

\[
\lim_{x \to \infty} f(x) = 1 \quad \lim_{x \to -\infty} f(x) = -1
\]

\( \rightarrow \) H.A.: \( y = -1, \ y = 1 \)
SLANT/OBLIQUE ASYMPTOTES

If the degree of the numerator is exactly one more than the degree of the denominator, then there is not a horizontal asymptote, but an oblique or slant asymptote. The equation is found by doing long division and the quotient is the equation of the slant or oblique asymptote ignoring the remainder.

\[ R(x) = \frac{x^3 + 2x^2 - 3x + 5}{x^2 - 3x + 4} \]

degree of top = 3

degree of bottom = 2

\[ x^2 - 3x - 4 \left[ x^3 + 2x^2 - 3x + 5 \right] \]

\[ x + 5 + \text{a remainder} \]

Slant asymptote at \( y = x + 5 \)
Limits of Rational Functions

Given that $f(x)$ is a polynomial of degree $m$ and $g(x)$ is a polynomial of degree $n$.

If $m > n$, then $\lim_{{x \to \infty}} \frac{f(x)}{g(x)} = \infty$

If $m < n$, then $\lim_{{x \to \infty}} \frac{f(x)}{g(x)} = 0$

If $m = n$, then $\lim_{{x \to \infty}} \frac{f(x)}{g(x)}$ is the ratio of the leading coefficients of $f$ and $g$. 
Continuous Functions
Definition of Continuity

A function $f$ is continuous at $x = a$ when:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exist
3. $\lim_{x \to a} f(x) = f(a)$

If any one of the conditions is not met, the function is not continuous at $x = a$. 
A Continuous Function Example

\[ y = 3x + 2 \]
Discontinuous Functions

\[ f(x) = \frac{x^2 - 1}{x - 1} \]

\[ x = 1 \] is not part of the domain
Derivatives and stuff....
What is a derivative?

- A function
- the rate of change of a function
- the slope of the line tangent to the curve
Derivative Formulas

In the following, \( u \) and \( v \) are functions of \( x \), and \( n, e, a, \) and \( k \) are constants.

1. \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \) The Definition of the Derivative.

2. \( \frac{d}{dx}(k) = 0 \) The derivative of a constant is zero.

3. \( \frac{d}{dx}(k(u(x))) = k \frac{du}{dx} \) The derivative of a constant times a function.

4. \( \frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx} \) The Power Rule (Variable raised to a constant).

5. \( \frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx} \) The Sum Rule.

6. \( \frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx} \) The Difference Rule.

7. \( \frac{d}{dx}(uv) = uv' + vu' \) The Product Rule.

8. \( \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \) The Quotient Rule.

9. \( \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \) The Chain Rule.
Linear Properties of Derivatives from the previous slide are the following:

3. \[ \frac{d}{dx} \left( k \cdot u(x) \right) = k \frac{du}{dx} \]  The derivative of a constant times a function.

5. \[ \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \]  The Sum Rule.

6. \[ \frac{d}{dx} (u - v) = \frac{du}{dx} - \frac{dv}{dx} \]  The Difference Rule.
Derivatives of Trig. Functions

\[
\begin{align*}
\frac{d}{dx} (\sin x) &= \cos x \\
\frac{d}{dx} (\cos x) &= -\sin x \\
\frac{d}{dx} (\tan x) &= (\sec x)^2 \\
\frac{d}{dx} (\sec x) &= \sec x \cdot \tan x \\
\frac{d}{dx} (\csc x) &= -\csc x \cdot \cot x \\
\frac{d}{dx} (\cot x) &= - (\csc x)^2
\end{align*}
\]
Derivatives of Exponential Functions

\[
\frac{d}{dx}(e^x) = e^x
\]

\[
\frac{d}{dx}(a^x) = a^x \cdot \ln a
\]

\[
\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0
\]

\[
\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \quad x \neq 0
\]

\[
\frac{d}{dx}(\log_a x) = \frac{1}{x \cdot \ln a}, \quad x > 0
\]
Additional Practice Problems

1. \( f(x) = (x-1)(x+1) \)

2. \( f(x) = (x^4 - 3)e^x \)

3. \( f(x) = \sin(x) \cdot \ln(x) \)

4. \( f(x) = (2-x^5)(3+x-x^2) \)

5. \( f(x) = x^2 \cdot \sqrt{x} \)
Answers:

1. \( f'(x) = 2x \)
2. \( f'(x) = e^x(x^4 + 4x^3 - 3) \)
3. \( f'(x) = \cos(x) \ln(x) + \frac{\sin(x)}{x} \)
4. \( f'(x) = 7x^6 - 6x^5 - 15x^4 - 4x + 2 \)
5. \( f'(x) = \frac{5}{2}x^\frac{3}{2} \)
Back to Derivatives
Derivative as a Rate of Change
Derivative as a Rate of Change

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<th>Higher Order Derivatives</th>
<th>Original Function (position /distance/ height)</th>
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<td>$y = f(x)$</td>
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<tr>
<td>$y' = f'(x) = \frac{dy}{dx}$</td>
<td>First Derivative (velocity)</td>
</tr>
<tr>
<td>$y'' = f''(x) = \frac{d^{2}y}{dx^{2}}$</td>
<td>Second Derivative (acceleration)</td>
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<tr>
<td>$y''' = f'''(x) = \frac{d^{3}y}{dx^{3}}$</td>
<td>Third Derivative (jerk)</td>
</tr>
<tr>
<td>$y^{(4)} = f^{(4)}(x) = \frac{d^{4}y}{dx^{4}}$</td>
<td>Fourth Derivative</td>
</tr>
<tr>
<td>etc.</td>
<td></td>
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</table>
Implicit Differentiation

1. Differentiate both sides of the equation with respect to $x$.
2. Apply the rules of differentiation. If an expression involves $y$ then include $\frac{dy}{dx}$
3. Isolate all the $\frac{dy}{dx}$ terms on one side of the equation.
4. Factor out $\frac{dy}{dx}$
5. Solve for $\frac{dy}{dx}$ by dividing.

**Example:** Find $\frac{dy}{dx}$ of $2x^2 - 3y^3 = 5$

$$\frac{d}{dx} \left[ 2x^2 - 3y^3 \right] = \frac{d}{dx} \left[ 5 \right]$$

$$4x - 9y^2 \frac{dy}{dx} = 0$$

$$4x - 9y^2 \frac{dy}{dx} = 0$$

$$4x - 9y^2 \frac{dy}{dx} = 0$$

$$4x - 9y^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{4x}{9y^2}$$

$$\frac{dy}{dx} = \frac{4x}{9y^2}$$
Practice:

- $x^2 + y^2 = 25$
- $3x^2 - 5y^2 + 9x = 25 - 15y$
Answers:

• \[ \frac{dy}{dx} = -\frac{x}{y} \]

• \[ \frac{6x+9}{10y-15} = \frac{dy}{dx} \]
Related Rates
What are related rates?

Related Rates

Definition:
When two or more related variables are changing with respect to time they are called related rates

- In related rate story problems, the idea is to find a rate of change (with respect to time) of one quantity by using the rate of change (with respect to time) of a related quantity.
Example

- If one leg AB of a right triangle increases at the rate of 2 inches per second, while the other leg AC decreases at 3 inches per second, find how fast the hypotenuse is changing when AB = 72 inches and AC = 96 inches.

Given:
\[
\frac{dz}{dt} = 2 \text{ in/sec} \quad \frac{dx}{dt} = -3 \text{ in/sec} \\
z = 72 \text{ in.} \quad x = 96 \text{ in.}
\]

Find \( \frac{dy}{dt} \)

\[
x \frac{dx}{dt} + z \frac{dz}{dt} = y \frac{dy}{dt}
\]

Plug in known values.

\[
x^2 + z^2 = y^2
\]

\[
\sqrt{(96)^2 + (72)^2} = y
\]

120 = y

What is y?
Example 5 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?
Solution We draw Figure 5 and let \( x \) be the distance from the man to the point on the path closest to the searchlight. We let \( \theta \) be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that \( dx/dt = 4 \text{ ft/s} \) and are asked to find \( d\theta/dt \) when \( x = 15 \). The equation that relates \( x \) and \( \theta \) can be written from Figure 5:

\[
\frac{x}{20} = \tan \theta \quad x = 20 \tan \theta
\]

Differentiating each side with respect to \( t \), we get

\[
\frac{dx}{dt} = 20 \sec^2 \theta \frac{d\theta}{dt}
\]

so

\[
\frac{d\theta}{dt} = \frac{1}{20} \cos^2 \theta \frac{dx}{dt}
\]

\[
= \frac{1}{20} \cos^2 \theta \cdot 4 = \frac{1}{5} \cos^2 \theta
\]

When \( x = 15 \), the length of the beam is 25, so \( \cos \theta = \frac{4}{5} \) and

\[
\frac{d\theta}{dt} = \frac{1}{5} \left( \frac{4}{5} \right)^2 = \frac{16}{125} = 0.128
\]

The searchlight is rotating at a rate of 0.128 rad/s.
Maximas/ Minimas and the Mean Value Theorem.
The 2\textsuperscript{nd} Derivative Test

The second derivative demonstrates whether a point with zero first derivative is a maximum, a minimum, or an inflexion point.

\[ \frac{dy}{dt} = 0 \]
\[ \frac{d^2y}{dt^2} < 0 \]

For a maximum, the second derivative is negative. The slope of the curve (first derivative) is at first positive, then goes through zero to become negative.

\[ \frac{dy}{dt} = 0 \]
\[ \frac{d^2y}{dt^2} > 0 \]

For a minimum, the second derivative is positive. The slope of the curve is first derivative is at first negative, then goes through zero to become positive.

\[ \frac{dy}{dt} = 0 \]
\[ \frac{d^2y}{dt^2} = 0 \]

For an inflexion point, the second derivative is zero at the same time the first derivative is zero. It represents a point where the curvature is changing its sense. Inflexion points are relatively rare in nature.
Extrema and the Second Derivative

$y' > 0$, the graph is going uphill.
$y' < 0$, the graph is going downhill.

$y' = 0$
- maximum, if $y' > 0$ to the left, $y' < 0$ to the right
- minimum, if $y' < 0$ to the left, $y' > 0$ to the right
- uphill flat point, if $y' > 0$ on both sides
- downhill flat point, if $y' < 0$ on both sides
Mean Value Theorem

If \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\)
then there is a \( c \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]
Graphing a $f(x)$ Using 1$^{st}$ and 2$^{nd}$ Derivatives
First, let's remember the basics:
Ideas to Graphing:

a) the domain 
b) intercepts 
c) equations of asymptotes (both vertical and horizontal) 
d) relative extrema (be sure to provide all derivatives, identify critical numbers, and show the test for those values – naming the test you are using.) 
e) intervals where the function is increasing or decreasing 
f) inflection points (be sure to identify possible inflection points and be sure to show the test for those values.) 
g) intervals where the graph is concave up and where it is concave down (be sure to include the vertical asymptotes in your intervals.) 
h) provide computer-generated graphs of your function that illustrate all of the key points that you have listed. You may need to provide more than one graph.
Let’s Practice!

Graph \( f(x) = -x^2(x + 2)(x - 3)^3 \)

2. \( x = -2, 0, 3 \)

First Impressions:
1. Polynomial, no restrictions on the domain \((-\infty, \infty)\)
2. What are the x/y-intercepts?
Set each piece equal to zero for x-intercepts.
3. Are there any asymptotes?
4. Take the 1st and 2nd Derivative
More Practice

\[ y = \frac{2x^3 + x^2 + 1}{x^2 + 1} \]

\[ y = \frac{2x^2}{x^2 - 1} \]
Optimization
The Optimization Problem is:

Find values of the variables that minimize or maximize the objective function while satisfying the constraints.
Components of Optimization Problem

Optimization problems are made up of three basic ingredients:

- An **objective function** which we want to minimize or maximize.
- A set of **unknowns** or **variables** which affect the value of the objective function.
- A set of **constraints** that allow the unknowns to take on certain values but exclude others.
Example 2: An open topped serving box will be made by cutting squares out of each corner of a 12" by 18" sheet of cardboard and folding the tabs up to form a box. What size squares should be cut out to maximize the volume of the box?

\[ V = (18-2x)(12-2x)x \]
\[ V = 216x - 60x^2 + 4x^3 \]
\[ V' = 216 - 120x + 12x^2 = 0 \]
\[ x^2 - 10x + 18 = 0 \]
\[ x = 2.35, 7.65 \]
Extreme Value Thm.
What is the Extreme Value Thm.?

**The Extreme Value Theorem** If \( f \) is continuous on a closed interval \([a, b]\), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) at some numbers \( c \) and \( d \) in \([a, b]\).

**The Closed Interval Method** To find the *absolute* maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b]\):

1. Find the values of \( f \) at the critical numbers of \( f \) in \((a, b)\).
2. Find the values of \( f \) at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.
Linear Approximation
What is linear approximation?

“Linear approximation, or linearization, is a method we can use to approximate the value of a function at a particular point. The reason linear approximation is useful is because it can be difficult to find the value of a function at a particular point.”
LINEAR APPROXIMATION

The linear function whose graph is this tangent line, shown, is called the Linearization of the function $f$ at $a$.

$$y = f(x)$$

$$(a, f(a))$$

$$y = L(x)$$

$$L(x) = f(a) + f'(a)(x - a)$$

[equation of tangent line]
Practice

1. $f(x) = 3x e^{2x-10}$ at $x = 5$

2. $h(t) = t^4 - 6t^3 + 3t - 7$ at $t = -3$
Reimann Sums
A Riemann sum is an approximation of a region's area, obtained by adding up the areas of multiple simplified slices of the region.
Riemann Sums Using Rules (Left - Right - Midpoint).

Consider a function \( f(x) \) defined on an interval \([a, b] \). The area under this curve is approximated by

\[
\sum_{i=1}^{n} f(c_i) \Delta x_i.
\]

1. When the \( n \) subintervals have equal length, \( \Delta x_i = \Delta x = \frac{b - a}{n} \).

2. The \( i^{th} \) term of the partition is \( x_i = a + (i - 1) \Delta x \). (This makes \( x_{n+1} = b \).)

3. The Left Hand Rule summation is: \( \sum_{i=1}^{n} f(x_i) \Delta x \).

4. The Right Hand Rule summation is: \( \sum_{i=1}^{n} f(x_{i+1}) \Delta x \).

5. The Midpoint Rule summation is: \( \sum_{i=1}^{n} f \left( \frac{x_i + x_{i+1}}{2} \right) \Delta x \).
Evaluate the integrals in terms of Riemann sum

8. \( \int_0^3 (x - 1) \, dx \)

9. \( \int_0^2 (2x - x^3) \, dx \)

10. \( \int_1^4 (x^2 - 4x + 2) \, dx \)
Definite Integrals
“The definite integral is defined to be exactly the limit and summation that we looked at in the last section to find the net area between a function and the x-axis. Also note that the notation for the definite integral is very similar to the notation for an indefinite integral.”
Properties of Definite Integral

Assuming $f$ and $g$ are continuous functions

\[
\int_{a}^{b} f(x) \, dx = \int_{b}^{a} f(x) \, dx
\]

\[
\int_{a}^{a} f(x) \, dx = 0
\]

\[
\int_{a}^{b} c \, dx = c(b - a), \text{ where } c \text{ is any constant}
\]

\[
\int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx, \text{ where } c \text{ is any constant}
\]

\[
\int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx
\]

\[
\int_{a}^{b} [f(x) - g(x)] \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx
\]

\[
\int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx = \int_{a}^{c} f(x) \, dx
\]
Trigonometric Antiderivatives:\n\[
\int \sin x \, dx = -\cos x + C
\]
\[
\int \cos x \, dx = \sin x + C
\]
\[
\int \sec^2 x \, dx = \tan x + C
\]
\[
\int \csc^2 x \, dx = -\cot x + C
\]
\[
\int \sec x \tan x \, dx = \sec x + C
\]
\[
\int \csc x \cot x \, dx = -\csc x + C
\]
Integrals - U substitution

**steps**

1. Pick U
2. Make "du" equal
3. Replace integral with U terms
4. Integrate
5. Replace answer with "x" terms

\[
\int 6x(\frac{x^2}{5})^5 \, dx
\]

**U = x - 5**

\[
\int U^5 (3du)
\]

\[
du = 2x \, dx
\]

\[
3du = 6x \, dx
\]
Practice

\[
\begin{align*}
(1) \quad & \int \sin x \cos^4 x \, dx \quad (2) \quad \int x^3 \cos^3 x \, dx \\
& - 1 \quad - \pi/4 \\
(3) \quad & \int_0^{\pi/2} \sin^3 x \cos x \, dx \\
& = 0 \\
(4) \quad & \int_{\pi/2}^{\pi/2} \cos^3 x \, dx \\
& = - \pi/2 \\
(5) \quad & \int_0^{\pi/2} \sin^2 x \cos x \, dx \\
& = - \pi/2 \\
(6) \quad & \int_{\pi/4}^{\pi/4} x \sin^2 x \, dx \\
& = - \pi/4
\end{align*}
\]
Fundamental Thm.
Of Calculus 1 & 2
Section 5.4 – Fundamental Theorem of Calculus

- Fundamental Theorem of Calculus, Part 1
  If $f$ is continuous on $[a, b]$, then the function
  
  $$F(x) = \int_{a}^{x} f(t) \, dt$$
  
  has a derivative at every point $x$ in $[a, b]$, and
  
  $$\frac{d}{dx} F(x) = \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x)$$

  Example: \( \frac{d}{dx} \int_{0}^{x} (\sin t) \, dt = \sin x \)
Fundamental Theorem of Calculus Part 2

If a function $f$ is continuous in the interval $[a,b]$ then the following is true.

$$\int_{a}^{b} f(x)dx = F(b) - F(a)$$

Here the function $F$ is the antiderivative of $f$, or $F' = f$
Area Between Curves
Area Between Two Curves

\[ A = \int_{a}^{b} [f(x) - g(x)] \, dx \]

\[ A = \int_{c}^{d} [f(y) - g(y)] \, dy \]
Volume of Solids of Revolution
Comparing the Methods for Finding the Volume of a Solid Revolution around the x-axis

<table>
<thead>
<tr>
<th>Compare</th>
<th>Disk Method</th>
<th>Washer Method</th>
<th>Shell Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume formula</td>
<td>$V = \int_a^b \pi [f(x)]^2 , dx$</td>
<td>$V = \int_a^b \pi [(f(x))^2 - (g(x))^2] , dx$</td>
<td>$V = \int_c^d 2\pi y , g(y) , dy$</td>
</tr>
<tr>
<td>Solid</td>
<td>No cavity in the center</td>
<td>Cavity in the center</td>
<td>With or without a cavity in the center</td>
</tr>
<tr>
<td>Interval to partition</td>
<td>$[a, b]$ on x-axis</td>
<td>$[a, b]$ on x-axis</td>
<td>$[c, d]$ on y-axis</td>
</tr>
<tr>
<td>Rectangle</td>
<td>Vertical</td>
<td>Vertical</td>
<td>Horizontal</td>
</tr>
</tbody>
</table>

Typical region

Typical element
Example 3  Use cylindrical shells to find the volume of the solid obtained by rotating about the $x$-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.
**Solution** This problem was solved using disks in Example 5.2.2. To use shells we relabel the curve \( y = \sqrt{x} \) (in the figure in that example) as \( x = y^2 \) in Figure 9. For rotation about the \( x \)-axis we see that a typical shell has radius \( y \), circumference \( 2\pi y \), and height \( 1 - y^2 \). So the volume is

\[
V = \int_0^1 (2\pi y)(1 - y^2) \, dy = 2\pi \int_0^1 (y - y^3) \, dy
\]

\[
= 2\pi \left[ \frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \frac{\pi}{2}
\]

In this problem the disk method was simpler.
Length of Curves
Arc Length

\[ L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx \]

\[ f'(x) = \frac{dy}{dx} \quad f'(y) = \frac{dx}{dy} \]

\[ L = \int_{c}^{d} \sqrt{1 + [f'(y)]^2} \, dy \]
EXAMPLE 4 Find the arc length function for the curve $y = x^2 - \frac{1}{8} \ln x$ taking $P_0(1, 1)$ as the starting point.
SOLUTION If \( f(x) = x^2 - \frac{1}{8} \ln x \), then

\[
f'(x) = 2x - \frac{1}{8x}
\]

\[
1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}
\]

\[
= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2
\]

\[
\sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x} \quad \text{(since } x > 0)\]

Thus the arc length function is given by

\[
s(x) = \int_1^x \sqrt{1 + [f'(t)]^2} \, dt
\]

\[
= \int_1^x \left(2t + \frac{1}{8t}\right) \, dt = t^2 + \frac{1}{8} \ln t \bigg|_1^x
\]

\[
= x^2 + \frac{1}{8} \ln x - 1
\]

For instance, the arc length along the curve from \((1, 1)\) to \((3, f(3))\) is

\[
s(3) = 3^2 + \frac{1}{8} \ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373
\]
What can you do before the semester starts:

<table>
<thead>
<tr>
<th>Mentality</th>
<th>Be proactive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Review</td>
<td>Review the self-evaluation</td>
</tr>
<tr>
<td>Explore</td>
<td>Explore online resources</td>
</tr>
<tr>
<td>Converse</td>
<td>Talk to your professor and TA</td>
</tr>
<tr>
<td>Locate</td>
<td>Find resources on campus, such as CAPS and tutoring</td>
</tr>
<tr>
<td>Study</td>
<td>Form a study group, develop a study plan</td>
</tr>
</tbody>
</table>
Throughout the semester

GO TO CLASS

STAY ON TOP OF HOMEWORK

GO TO PROFESSOR AND TA OFFICE HOURS, CAPS, CALC TABLE.
goto.unm.edu/ess-feedback

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Questions?